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Conjugate Gradient Methods Applied to Transonic Finite Difference and Finite Element Calculations

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Most transonic finite difference and finite element calculations are obtained by SLOR. Recently, approximate factorization methods (ADI, AF2, SIP) have been used with finite differences (application of such iterative methods to finite elements is not straightforward). In this paper incomplete LU decomposition and SSOR are used as preconditioning to second-degree iterative methods with adaptive acceleration parameters as in conjugate gradient algorithms for both finite difference and finite element calculations based on the artificial density formulation. Different cases are tested and the results are demonstrated. The present method is certainly more efficient than pure SLOR for obtaining results with reasonable accuracy.

I. Introduction

RECENTLY, the method of conjugate gradients (CG) has been used successfully to accelerate the convergence of basic iterative procedures designed to solve boundary-value problems. The use of preconditioned CG methods for elliptic problems which lead to symmetric positive definite systems has been widely accepted since the work of Axelsson,[1] Concus et al.,[2] Meijerink and van der Vorst,[3] and Kershaw.[4] Some attempts have been made to generalize such procedures to nonsymmetric and/or indefinite systems[5-9] with less success.

The purpose of this paper is to apply the CG method to transonic potential flow calculations. The governing equation is nonlinear and of mixed type and admits a discontinuous solution. However, with the artificial compressibility method[10-12] (where the density is slightly modified to account for the artificial viscosity needed in the supersonic zone), the equation looks elliptic, assuming the density is known from the previous iteration. For both finite differences and finite elements, the resulting algebraic equations are solved by successive line over-relaxation (SLOR) marching with the flow direction.

Because of the slow rate of convergence, SLOR is replaced by approximate factorization methods for finite difference calculations. For finite elements, ADI-type methods are not easy to implement[13,14] since the matrix equation, in general, is more complicated. It is shown here that incomplete LU decomposition is a simple and effective approximate factorization, which works well for both finite differences and finite elements.

Moreover, in SLOR and ADI-type methods, the rate of convergence depends on the choice of a parameter or a sequence of parameters and the optimum choice is usually obtained by experiments. In CG methods, the parameters are calculated in terms of the iterative solution as a part of the algorithm. Compared to other acceleration procedures, CG methods are more reliable than the extrapolation method,[15] and simpler than the multigrid method.[16]

In the following, the continuous problem is formulated and different discretization and iterative procedures are discussed. Numerical results are shown and the effectiveness of the present method is demonstrated.

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II. Problem Formulation

Inviscid isentropic flows are governed by a kinematical relation, namely conservation of mass, and in terms of a velocity potential ϕ, the governing equation is

$$\nabla \cdot \rho \vec{q} = 0 \tag{1}$$

where

$$\vec{q} = \nabla \phi, \quad \rho = (M_\infty^2 a^2)^{\frac{1}{\gamma-1}}, \quad a^2 = \frac{1}{M_\infty^2} - \frac{\gamma-1}{2}(q^2-1)$$

Here ρ is the fluid density, γ the ratio of specific heats, and M_∞ the freestream Mach number. The tangential and wake boundary conditions and the requirement that the velocity vanishes at infinity completes the formulation.

Using the Gauss theorem, the integral formulation is

$$\iint \nabla \cdot \rho \vec{q} \mathrm{d}A = \int \rho \vec{q} \cdot n \mathrm{d}s = 0 \tag{2}$$

The variational formulation is given by the Bateman principle, namely,

$$I = \iint p \mathrm{d}A \tag{3}$$

is stationary, where $p = \rho^\gamma / \gamma M_\infty^2$. I has an extremum for subsonic flows only.

The three formulations are, of course, equivalent. For example, the Euler equation of the functional (3), for two-dimensional flows, using Cartesian coordinates is

$$\left(\frac{\partial p}{\partial u}\right)_x + \left(\frac{\partial p}{\partial v}\right)_y = 0 \tag{4}$$

where $u = \phi_x$ and $v = \phi_y$. In terms of ϕ, Eq. (4) becomes

$$(\rho \phi_x)_x + (\rho \phi_y)_y = 0 \tag{5}$$

There are natural boundary conditions associated with Eq. (3) across the body, the shock, and the wake.

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_B = \frac{\phi_y}{\phi_x} \tag{6a}$$

$$\left(\frac{dy}{dx}\right)_S = \frac{[\![\rho\phi_y]\!]}{[\![\rho\phi_x]\!]} \tag{6b}$$

$$\left(\frac{dy}{dx}\right)_W = \frac{[\![\phi_y]\!]}{[\![\phi_x]\!]} \tag{6c}$$

where $[\![\phi_x]\!]_S$ denotes the jump in ϕ_x across the shock. The solution ϕ can be represented as a linear combination of trial (shape) functions. If the trial functions do not satisfy the boundary conditions, the functional (3) must be modified accordingly.

In Ref. 17, I is replaced by a functional of two variables ρ and ϕ

$$I(\phi,\rho) = \iint [\rho(\phi_x^2+\phi_y^2)+F(\rho)]\,dA \tag{7}$$

Hence, $I_\phi = 0$ gives

$$(\rho\phi_x)_x + (\rho\phi_y)_y = 0 \tag{8a}$$

while $I_\rho = 0$ gives

$$\rho = \phi_x^2 + \phi_y^2 + F'(\rho) \tag{8b}$$

$F(\rho)$ is chosen so that Eq. (8b) is consistent with Bernoulli's equation. Thus, a generalized gradient method, based on Eq. (7), is

$$\begin{bmatrix} \delta\phi \\ \delta\rho \end{bmatrix} = -C \begin{bmatrix} I_\phi \\ I_\rho \end{bmatrix} \tag{9}$$

where $\delta\phi$ and $\delta\rho$ are the correction to ϕ and ρ, respectively. The elements of the matrix C can be chosen such that Eq. (9) represents the physical unsteady flow

$$\rho_t = -(\rho u)_x - (\rho v)_y \tag{10a}$$

$$\phi_t = (1-\rho^{\gamma-1})/(\gamma-1)M_\infty^2 - (\phi_x^2+\phi_y^2)/2 \tag{10b}$$

It should be mentioned that if ϕ_t is set to zero in Eq. (10b), ρ_t can be evaluated in terms of ϕ as follows.

$$\rho_t = -(\rho/a^2)uu_t - (\rho/a^2)vv_t \tag{11}$$

The right-hand side of Eq. (11) is proportional to the ϕ_{st} term needed for convergence of most iterative procedures when applied to transonic flow problems, as will be discussed later.

Any of the preceding formulation admits expansion as well as compression shocks. To exclude the unphysical solution, an artificial viscosity is added by modifying the density as follows.

$$\tilde{\rho} = \rho - \mu\rho_s\Delta s \tag{12}$$

where

$$\mu = \max[0, 1-(1/M^2)]$$

and

$$\rho_s\Delta s \simeq (u/q)\rho_x\Delta x + (v/q)\rho_y\Delta y$$

Equation (12) can be interpreted as an approximation of

$$\tilde{\rho} \simeq \rho^{-\mu\rho_s\Delta s/\rho} \tag{13}$$

where $\mu\rho_s\Delta s/\rho$ is an artificial entropy.

III. Discretization Techniques

Usually, finite differences are associated with Taylor series, finite volumes with the Gauss theorem, and finite elements with a variational formulation. The three discretization techniques are applied here to the full potential equation, Eq. (5). Let

$$\mathrm{DXE} = x_{i+1} - x_i \qquad \mathrm{DXW} = x_i - x_{i-1}$$

$$\mathrm{DYN} = y_{j+1} - y_j \qquad \mathrm{DYS} = y_j - y_{j-1}$$

Assume the density is constant in each cell,

$$\mathrm{RNE} = \tilde{\rho}_{i+\frac12,j+\frac12} \qquad \mathrm{RSE} = \tilde{\rho}_{i+\frac12,j-\frac12}$$

$$\mathrm{RNW} = \tilde{\rho}_{i-\frac12,j+\frac12} \qquad \mathrm{RSW} = \tilde{\rho}_{i-\frac12,j-\frac12}$$

The finite difference approximation is (see Fig. 1a)

$$\mathrm{CN}\phi_{i,j+1} + \mathrm{CW}\phi_{i-1,j} + \mathrm{CS}\phi_{i,j-1} + \mathrm{CE}\phi_{i+1,j} + \mathrm{CC}\phi_{i,j} = 0$$

where

$$\begin{aligned}
&\mathrm{CN} = \mathrm{RNE}^*(\mathrm{DXE}/\mathrm{DYN}) + \mathrm{RNW}^*(\mathrm{DXW}/\mathrm{DYN})\\
&\mathrm{CW} = \mathrm{RNW}^*(\mathrm{DYN}/\mathrm{DXW}) + \mathrm{RSW}^*(\mathrm{DYS}/\mathrm{DXW})\\
&\mathrm{CS} = \mathrm{RSW}^*(\mathrm{DXW}/\mathrm{DYS}) + \mathrm{RSE}^*(\mathrm{DXE}/\mathrm{DYS})\\
&\mathrm{CE} = \mathrm{RNE}^*(\mathrm{DYN}/\mathrm{DXE}) + \mathrm{RSE}^*(\mathrm{DYS}/\mathrm{DXE})\\
&\mathrm{CC} = -\mathrm{CN} - \mathrm{CW} - \mathrm{CS} - \mathrm{CE}
\end{aligned} \tag{14}$$

and the finite volume approximation is (see Fig. 1b)

$$\begin{aligned}
&\mathrm{CNE}\phi_{i+1,j+1} + \mathrm{CNW}\phi_{i-1,j+1} + \mathrm{CSW}\phi_{i-1,j-1} + \mathrm{CSE}\phi_{i-1,j-1}\\
&\quad + \mathrm{CC}\phi_{i,j} + \mathrm{CCN}\phi_{i,j+1} + \mathrm{CCW}\phi_{i-1,j} + \mathrm{CCS}\phi_{i,j-1}\\
&\quad + \mathrm{CCE}\phi_{i+1,j} = 0
\end{aligned}$$

where

$$\begin{aligned}
&\mathrm{CNE} = \mathrm{RNE}^*(\mathrm{DXE}/\mathrm{DYN} + \mathrm{DYN}/\mathrm{DXE})\\
&\mathrm{CNW} = \mathrm{RNW}^*(\mathrm{DXW}/\mathrm{DYN} + \mathrm{DYN}/\mathrm{DXW})\\
&\mathrm{CSW} = \mathrm{RSW}^*(\mathrm{DXW}/\mathrm{DYS} + \mathrm{DYS}/\mathrm{DXW})\\
&\mathrm{CSE} = \mathrm{RSE}^*(\mathrm{DXE}/\mathrm{DYS} + \mathrm{DYS}/\mathrm{DXE})\\
&\mathrm{CCN} = 2^*\mathrm{CN} - \mathrm{CNW} - \mathrm{CNE}\\
&\mathrm{CCW} = 2^*\mathrm{CW} - \mathrm{CNW} - \mathrm{CSW}\\
&\mathrm{CCS} = 2^*\mathrm{CS} - \mathrm{CSW} - \mathrm{CSE}\\
&\mathrm{CCE} = 2^*\mathrm{CE} - \mathrm{CSE} - \mathrm{CNE}
\end{aligned} \tag{15}$$

For equal meshes CCN = CCW = CCS = CCE = 0; and the odd and the even points are decoupled. Hence, finite volumes can not be used unless they are modified to avoid the decoupling problem.

A variational formulation leads to the finite difference formula Eq. (12) if the functional

$$I=\iint \rho\,(\phi_x^2+\phi_y^2)\,\mathrm{d}A$$

is approximated by $I=I_{\rm NE}+I_{\rm NW}+I_{\rm SW}+I_{\rm SE}$, where

$$I_{\rm NW}=\tfrac{1}{2}\mathrm{RNE^*DXE^*DYN^*}\left[\left(\frac{\phi_{i+1,j}-\phi_{i,j}}{\mathrm{DXE}}\right)^2+\left(\frac{\phi_{i+1,j+1}-\phi_{i,j+1}}{\mathrm{DXE}}\right)^2+\left(\frac{\phi_{i+1,j+1}-\phi_{i+1,j}}{\mathrm{DYN}}\right)^2+\left(\frac{\phi_{i,j+1}-\phi_{i,j}}{\mathrm{DYN}}\right)^2\right] \tag{16}$$

with similar expressions for the other cells; while the approximation

$$I_{\rm NW}=\tfrac{1}{2}\mathrm{RNE^*DXE^*DYN^*}\left\{\left[\tfrac{1}{2}\left(\frac{\phi_{i+1,j}-\phi_{i,j}}{\mathrm{DXE}}+\frac{\phi_{i+1,j+1}-\phi_{i,j+1}}{\mathrm{DXE}}\right)\right]^2+\left[\tfrac{1}{2}\left(\frac{\phi_{i+1,j+1}-\phi_{i+1,j}}{\mathrm{DYN}}+\frac{\phi_{i,j+1}-\phi_{i,j}}{\mathrm{DYN}}\right)\right]^2\right\} \tag{17}$$

and similar expressions for the other cells, lead to the finite volume formula, Eq. (15).

The finite element formula is, however, different from both. If a bilinear element is used, for example, ϕ in the north east cell is approximated by

$$\phi=a+bx+cy+\mathrm{d}xy \tag{18}$$

and, hence,

$$I_{\rm NW}=[(\mathrm{RNE^*DXE^*DYN})/3]\left[\left(\frac{\phi_{i+1,j}-\phi_{i,j}}{\mathrm{DXE}}\right)^2+\left(\frac{\phi_{i+1,j+1}-\phi_{i,j+1}}{\mathrm{DXE}}\right)^2+\left(\frac{\phi_{i+1}-\phi_{i,j}}{\mathrm{DXE}}\right)\left(\frac{\phi_{i+1,j+1}-\phi_{i,j+1}}{\mathrm{DXE}}\right)+\left(\frac{\phi_{i+1,j+1}-\phi_{i+1,j}}{\mathrm{DYN}}\right)^2+\left(\frac{\phi_{i,j+1}-\phi_{i,j}}{\mathrm{DYN}}\right)^2+\left(\frac{\phi_{i+1,j+1}-\phi_{i+1,j}}{\mathrm{DYN}}\right)\left(\frac{\phi_{i,j+1}-\phi_{i,j}}{\mathrm{DYN}}\right)\right] \tag{19}$$

The finite element formula is

$$(\mathrm{CN}+\mathrm{CCN})\phi_{i,j+1}+\mathrm{CNW}\phi_{i-1,j+1}+(\mathrm{CW}+\mathrm{CCW})\phi_{i-1,j}+\mathrm{CSE}\phi_{i-1,j-1}+(\mathrm{CS}+\mathrm{CCS})\phi_{i,j+1}+\mathrm{CSW}\phi_{i+1,j-1}+(\mathrm{CE}+\mathrm{CCE})\phi_{i+1,j}+\mathrm{CNE}\phi_{i+1,j+1}+2\mathrm{CC}\phi_{i,j}=0 \tag{20}$$

Identical results are obtained by applying the Galerkin method to Eq. (5). The finite element stencil is shown in Fig. 1c.

The finite element stencil is the sum of one finite difference and one finite volume stencil as noted in Ref. 18.

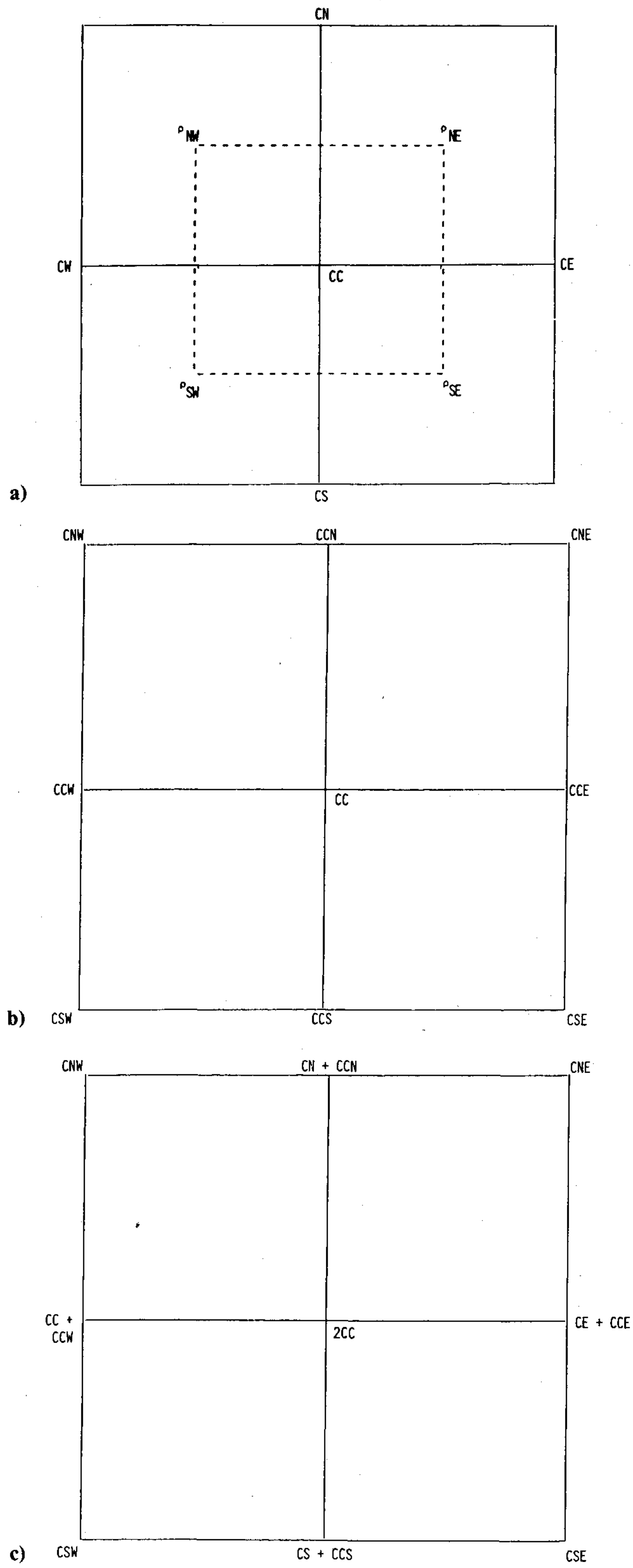

Fig. 1 a) Finite difference stencil. b) Finite volume stencil. c) Finite element stencil.

Treatment of Boundary Conditions

The exact Neumann condition $\phi_n=0$ is implemented easily, for example, if the points $(i-1,j)$, (i,j), and $(i+1,j)$ lie on the boundary, and RSE and RSW are set to equal zero. In the case of the linearized boundary condition $\phi_y=f'(x)$, the functional has to be modified by adding the terms $-2\int f'(x)\phi\,\mathrm{d}x$. The condition from the north east cell is

$$-(\mathrm{DXE}/3)^*\,[2f_i'+f_{i+1}']$$

with similar contribution from the north west cell.

It can be shown that for unequal grids, the finite difference formula is first-order accurate while the finite element formula is second order.

In general, an isoparametric element is used with a local mapping which has the same form of the potential ϕ, and the nodal equations are assembled automatically.

IV. Iterative Procedures

The discretization techniques reduce the continuous problem to a system of N nonlinear algebraic equations (N is the number of unknowns)

$$F_\ell(\phi_{i,j})=0 \qquad \ell=1,\ldots,N \tag{21a}$$

Assuming the density is known, Eq. (21a) can be written in the form

$$A\phi=b \tag{21b}$$

where A is a symmetric positive definite matrix. In the case of finite differences, A has five nonzero diagonals while the finite element matrix has nine nonzero diagonals. In this section, the preconditioned conjugate gradient method used to solve Eq. (21b) is briefly reviewed.

Let,

$$r_0=b-A\phi_0, \qquad Cz_0=r_0, \qquad p_0=z_0$$

where r is the residual vector, p the direction vector, and C a matrix operator which approximates A. C can be a Laplacian operator or an incomplete LU factorization of A. Then for $n=0,1,2,\ldots$, compute

I)

$$\phi_{n+1}=\phi_n+\alpha_n p_n \tag{22a}$$

where

$$\alpha_n=(r_n,z_n)/(p_n,Ap_n)$$

II)

$$r_{n+1}=r_n-\alpha_n Ap_n \tag{22b}$$

III)

$$Cz_{n+1}=r_{n+1} \tag{22c}$$

IV)

$$p_{n+1}=z_{n+1}+\beta_n p_n \tag{22d}$$

where

$$\beta_n=(r_{n+1},z_{n+1})/(r_n,z_n)$$

A system of linear equations is solved in each cycle and five arrays are stored ($\phi, r, p, Ap,$ and z). The key to the recent success of the CG method is the choice of a proper scaling. Two choices of C are used in this study.

1) Preconditioning by incomplete factorization:

$$C=LL^T \tag{23}$$

where L denotes a lower triangular matrix. Unlike the exact factorization, L has nonzero entries only in those positions which correspond to nonzero elements in the lower triangle of A. The elements of L are computed by the row-sum agreement algorithm.[19,20]

Let,

$$A\phi_{i,j}=c_{i,j}\phi_{i+1,j}+f_{i,j}\phi_{i,j+1}+b_{i,j}\phi_{i,j}+c_{i-1,j}\phi_{i-1,j}+f_{i,j-1}\phi_{i,j-1}$$

Hence,

$$L\phi_{i,j}=v_{i,j}\phi_{i,j}+t_{i-1,j}\phi_{i-1,j}+g_{i,j-1}\phi_{i,j-1}$$

It can be shown that the product of LL^T has seven nonzero diagonals with two extra nonzeros appearing in the $(i+1,j-1)$ and $(i-1,j+1)$ positions. The general algorithm is given as follows.

$$v_{i,j}=(b_{i,j}-g_{i,j-1}^2-t_{i-1,j}^2-t_{i-1,j}g_{i-1,j}-t_{i,j-1}g_{i,j-1})^{1/2}$$
$$t_{i,j}=c_{i,j}/v_{i,j}, \qquad g_{i,j}=f_{i,j}/v_{i,j} \tag{24}$$

Note that the off-diagonal elements of LL^T whose location corresponds to the nonzero off-diagonal elements of A are equal to those of A. Moreover, the row sums of LL^T are the same as those of A. This algorithm can be extended to nonsymmetric matrices and to a matrix with nine nonzero diagonals as in finite elements.

2) Preconditioning by symmetric successive overrelaxation (SSOR): SSOR is a two-step iteration process; the first step is identical to SOR and the second is another SOR in which the equations are taken in a reverse order. If A is written in the form

$$A=D-\tilde{L}-\tilde{L}^T$$

where D contains the diagonal elements of A, and $\tilde{L}$ is a strictly low triangular then,

$$C=\frac{1}{\omega(2-\omega)}(D-\omega\tilde{L})D^{-1}(D-\omega\tilde{L}^T) \tag{25}$$

The rate of convergence of the overall method is not as sensitive to the choice of ω as SOR.

The preceding preconditioned CG methods, based on Eq. (23) or (25), can be described as a second degree method

$$(\phi_{n+2}-2\phi_{n+1}+\phi_n)+\left(1-\frac{\alpha_{n+1}}{\alpha_n}\beta_n\right)(\phi_{n+1}-\phi_n)$$
$$=\alpha_{n+1}Cr_{n+1}$$

Note that such methods converge for subsonic flows only. Applications for transonic flows will be discussed next.

V. Application of Iterative Methods to Transonic Flow Calculations

The most successful transonic full potential calculations were made by Jameson.[22] The transonic equation is first rewritten in nonconservative form

$$(\rho/a^2)[(a^2-u^2)\phi_{xx}-2uv\phi_{xy}+(a^2-v^2)\phi_{yy}]=0 \tag{26}$$

Equation (26) is rearranged in the form

$$(1-M^2)\phi_{ss}+\phi_{nn}=0 \tag{27}$$

where

$$\phi_{ss}=\frac{u^2}{q^2}\phi_{xx}+\frac{2uv}{q^2}\phi_{xy}+\frac{v^2}{q^2}\phi_{yy}$$
$$\phi_{nn}=\frac{v^2}{q^2}\phi_{xx}-\frac{2uv}{q^2}\phi_{xy}+\frac{u^2}{q^2}\phi_{yy}$$

∂_s and ∂_n can be interpreted as the derivative along the streamline direction and normal to it.

Equation (26) is reduced to a system of algebraic equations via Jameson's rotated difference scheme. Unlike Eq. (21b), the system matrix is not symmetric. Solutions are obtained by SLOR marching with the flow direction. A similar procedure is used for conservative calculations except the residual is written in conservation form. The same nonconservative algorithm is used, and mass is conserved only when the solution converges.

Jameson analyzed the convergence of SLOR as applied to Eq. (26). Here, his analysis is generalized to three level iterative schemes.[22]

Consider

$$\epsilon\phi_t + \gamma\phi_{tt} + 2\alpha\phi_{st} + 2\beta\phi_{nt} = (a^2 - q^2)\phi_{xx} + a^2\phi_{nn} \tag{28}$$

and let

$$T = t + [\alpha/(a^2 - q^2)]s + (\beta/a^2)n, \qquad S = s, \qquad N = n$$

Equation (28) becomes

$$\epsilon\phi_T + \left(\gamma + \frac{\alpha^2}{a^2 - q^2} + \frac{\alpha^2}{a^2}\right)\phi_{TT} = (a^2 - q^2)\phi_{SS} + a^2\phi_{NN} \tag{29}$$

For supersonic flows, the coefficient of ϕ_{TT} must be negative in order to have a hyperbolic equation, where S is the time-like direction, i.e.,

$$\alpha^2 > (M^2 - 1)\beta + \gamma(q^2 - a^2) \tag{30}$$

Also, the sign of α has to be chosen to guarantee the right domain of dependence. Moreover, since t is no longer the time-like direction, $\epsilon\phi_t$ does not present a proper damping in the supersonic zone and, therefore, ϵ must vanish there. A von Neumann test confirms this observation.[22] Equivalently, Eq. (28) can be written as a system of first-order equations

$$\begin{bmatrix} W \\ u \\ v \end{bmatrix}_t = A\begin{bmatrix} W \\ u \\ v \end{bmatrix}_x + B\begin{bmatrix} W \\ u \\ v \end{bmatrix}_y + C\begin{bmatrix} W \\ u \\ v \end{bmatrix} \tag{31}$$

where

$$W = \phi_t, \qquad u = \phi_x, \qquad v = \phi_y$$

The condition that this system is hyperbolic is given in terms of λ defined by

$$|A + \mu B - \lambda I| = 0 \tag{32}$$

where, for any real μ, λ is real. This condition leads to Eq. (30).

The preceding analysis indicates that the ϕ_{st} term is indispensable and it cannot be replaced by a ϕ_{tt} term. The one-dimensional example is simple and clear. Let

$$\gamma\phi_{tt} + 2\alpha u\phi_{xt} = (a^2 - u^2)\phi_{xx} \tag{33}$$

The characteristics of Eq. (33) are

$$\lambda_{1,2} = [-\alpha u \pm \sqrt{a^2u^2 + (a^2 - u^2)\gamma}]/\gamma \tag{34}$$

The limit when $\gamma \to 0$ is

$$\gamma_{1,2} = (a^2 - u^2)/2\alpha u, \infty \tag{35}$$

However, if α vanishes, the characteristics are not necessarily real, hence, the ϕ_{xt} term is essential for two level as well as three level schemes [α is chosen such that $\alpha^2u^2 + (a^2 - u^2)\gamma > 0$].

The considerations just given explain why SLOR converges for transonic flow problems if the marching direction is aligned with the flow direction since then there is a ϕ_{st} term in the equation describing the iterative process. Other methods (e.g., conjugate gradients) do not have this feature and consequently fail to converge. The main conclusion is that the transonic problem is not symmetric and the iterative procedure must be modified accordingly. For example, a procedure described by

$$A(\rho_n)\phi_{n+1} = b \tag{36}$$

where at each iteration, a symmetric matrix is inverted, works only for subsonic flows.

In the following, different approaches to handle the nonsymmetric nature of the transonic problem are discussed.

Generalized Least Squares Formulation

Given the differential equation $D\phi = f$, an associated functional whose second variation is positive definite can always be formulated using least squares, namely, minimizing the residual

$$\|D\phi - f\| = (D\phi, D\phi) - 2(f, D\phi) + (f,f) = (D^*D\phi, \phi)$$
$$-2(D^*f, \phi) + (f,f) \tag{37}$$

where the usual notations of norm, inner products, and adjoints are used. We notice that

$$I = (D^*D\phi, \phi) - 2(D^*f, \phi)$$

is the Ritz variational functional for the normal problem $D^*D\phi = D^*f$, which is automatically self-adjoint.

If D is replaced by Eq. (5), which is equivalent to Eq. (26), except in the shock region, D^* is given by

$$D^* = \left(\frac{\rho}{a^2}(a^2 - u^2)\frac{\partial}{\partial x}\right)_x - \left(\frac{\rho}{a^2}uv\frac{\partial}{\partial y}\right)_x$$
$$+ \left(\frac{\rho}{a^2}(a^2 - v^2)\frac{\partial}{\partial y}\right)_y - \left(\frac{\rho}{a^2}uv\frac{\partial}{\partial x}\right)_y \tag{38}$$

Using the Laplacian as a preconditioning operator[17] the problem becomes

$$C^{-1}D^*C^{-1}(D\phi - f) = 0 \tag{39}$$

Applying the gradient method to Eq. (39) gives

$$\delta\phi = -\tau C^{-1}D^*C^{-1}(D\phi - f) \tag{40}$$

which may be rewritten in the factored form

$$Cz = D\phi - f, \qquad C\delta\phi = -\tau D^*z$$

Glowinski et al.[23] used an equivalent procedure with finite elements where both D and D^* are modified due to an artificial viscosity term.

A conjugate gradient method with incomplete LU decomposition is applied to the normal equation where D is replaced by the small disturbance equation discretized via Murman's operators as will be discussed next. The method is not attractive since D^* has to be recalculated frequently.

Combined Iterations

Here, SLOR is used as preconditioning. This idea is implemented in a simple way. One iteration of SLOR is followed by an application of the conjugate gradient algorithm scaled with the Laplacian (thus, LL^T is used once and for all). The combined iteration idea is used by Jameson[21] with fast solvers as an alternative to other methods where the scaling matrix is desymmetrized. In comparing conjugate gradients with fast solvers, one should mention that there is no restriction on the grid in any direction and the CG method holds promise for three-dimensional calculations as well.[19] At any rate, there is no need to invert the matrix and get an exact solution for the correction at each iteration as in the case for fast solvers since an approximate correction at each iteration will suffice as long as the overall process converges.

Finally, it should be mentioned that the CG method is easily vectorizable and has been tested with other relaxation methods such as zebra and zebroid[22] (where alternating horizontal lines are solved simultaneously; explicitly by the zebra scheme, or implicitly by zebroid using the Thomas algorithm).

VI. Numerical Results

Results of transonic potential calculations around a cylinder and NACA 0012 airfoil at zero angle of attack are described. Particular attention is focused on the advantages of the present combined relaxation plus CG method over standard relaxation methods. The rate of convergence and the CPU time in seconds on the CYBER-175 computer are compared for different Mach numbers. In all cases, the maximum residual (R_{max}) is used to measure the convergence of the iterative process. The initial potential is set to zero for all calculations.

Two-Dimensional Small Disturbance Calculations

Using both Murman and Cole type dependent differencing,[24] and Murman's fully conservative operators,[25] the nonlinear transonic small disturbance equation (TSDE) is reduced to an unsymmetric system of equations. The conjugate gradient method is applied to the normal equation with preconditioning matrices based on an incomplete LU decomposition of the unsymmetric matrix and its conjugate. The computational results show that the CG method is not competitive to the combined iterations. It should be mentioned, however, that Khosla and Rubin[26] solved the normal equation by the CG method in a similar way but only subsonic results were presented.

Two-Dimensional Full Potential Calculations

Finite Difference Calculation around a Circular Cylinder

SLOR and combined SLOR + CG methods are compared. A 61×31 grid is used in all cases with uniform mesh in θ direction and a 15% stretching in the r direction.

Figures 2a and 2b show the rates of convergence of the two methods for various Mach numbers. A saving in computing time by at least a factor of 2 is achieved for supercritical cases. The pressure distributions are plotted in Fig. 3.

One of the features of the present method is that the convergence rate is not as sensitive to the relaxation parameter ω as in SLOR. The rate of convergence of the combined SLOR + CG method is almost constant for a wide range of ω as demonstrated in Fig. 4.

Figure 5 shows the convergence rates for a 101×41 grid. The system of equations is roughly doubled. Comparing Fig. 5 to Fig. 2b it is clear that the effectiveness of the combined method increases as the mesh is refined.

Two different preconditioning matrices are tested. One is based on ρC where ρ is the nodal density and C is an incomplete LL^T decomposition of the Laplacian operator, and the other is based on SSOR. Their performance for various Mach numbers is shown in Fig. 6. LL^T provides faster convergence rates for all problems tested.

Finite Difference and Finite Element Calculations around NACA 0012 Airfoil

Rates of convergence of SLOR, as well as SLOR + CG, for finite difference calculations are presented in Fig. 7. The

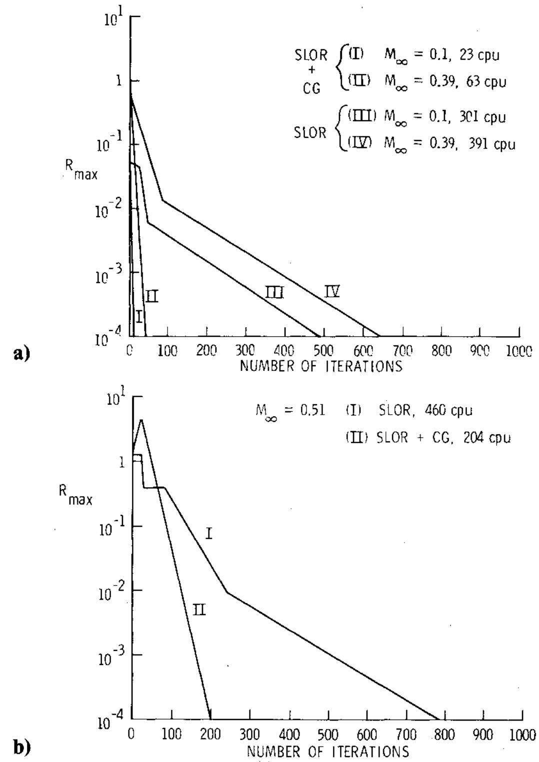

Fig. 2 Rates of convergence of finite difference calculations around a circular cylinder.

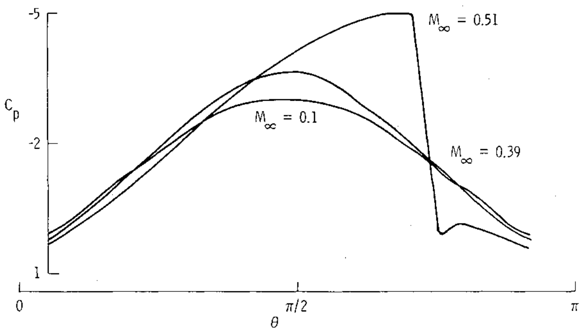

Fig. 3 Pressure distribution around a cylinder at different Mach numbers.

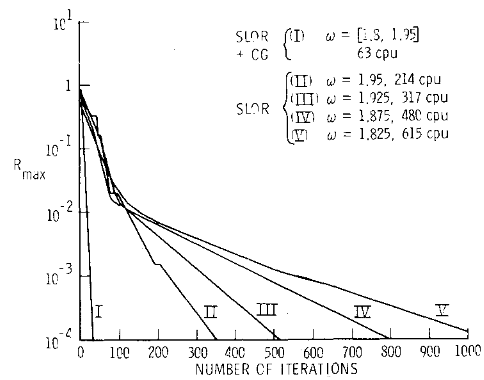

Fig. 4 Dependence of convergence rates on relaxation parameter ω at $M_\infty = 0.39$.

corresponding results for finite element calculations are shown in Fig. 8. The combined relaxation conjugate gradient methods are certainly superior to relaxation methods alone for subsonic flows. For supercritical flows the combined iterations provide a savings in computer time of up to a factor of three for both finite difference and finite element

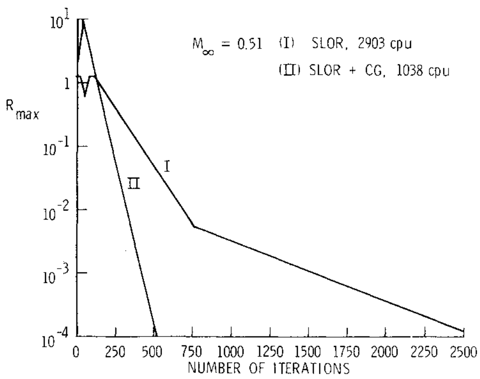

Fig. 5 Rates of convergence for a finer grid at $M_\infty = 0.51$.

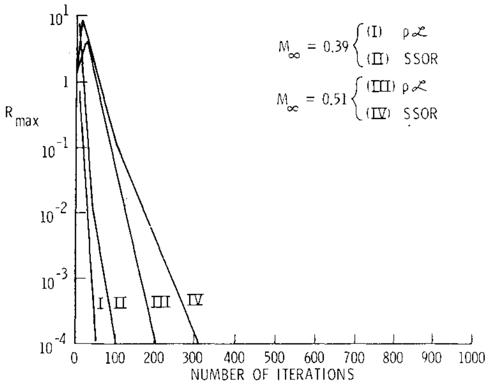

Fig. 6 Comparison of SSOR and ρLL^T preconditioning for subcritical and supercritical flows.

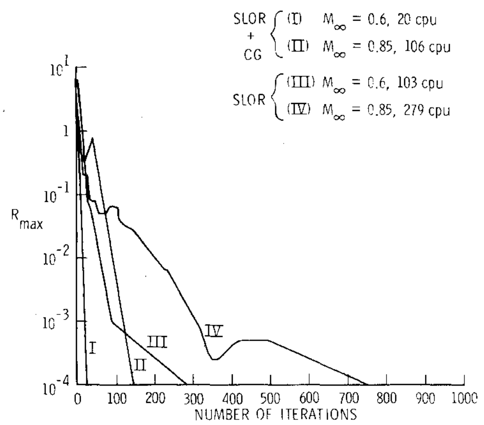

Fig. 7 Rates of convergence of finite difference calculations around NACA 0012 airfoil.

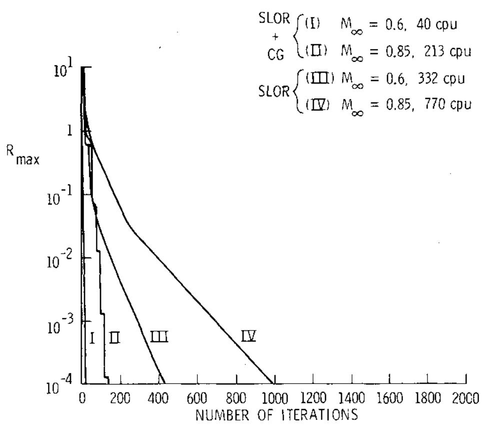

Fig. 8 Rates of convergence of finite element calculations around NACA 0012 airfoil.

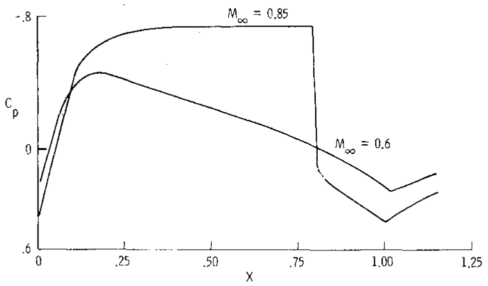

Fig. 9 Pressure distribution around NACA 0012 airfoil for different Mach numbers.

calculations. These savings are even greater if finer grids are used.

Numerical results show that while the convergence rates of relaxation methods are strongly influenced by using stretched grids, the present method is relatively insensitive to various stretchings in both x and y directions.

Figure 9 shows the pressure distribution on the airfoil for different Mach numbers.

VII. Concluding Remarks

Finite difference, finite volume, and finite element discretizations of the transonic full potential boundary-value problem are given based on the differential, integral, and variational formulation, respectively.

Finite difference and finite element calculations around an airfoil and a cylinder are computed by preconditioned conjugate gradient algorithms.

Different approaches have been tested. Generalized least squares are time consuming. On the other hand, a combined iteration procedure where SLOR is followed by a conjugate gradient algorithm preconditioned by an incomplete approximate factorization of a Laplacian proves to be efficient. Savings of CPU times of a factor of 10 for subsonic flows and of at least a factor of 2 for tough transonic cases are obtained, compared to existing calculations with SLOR only. Unlike SLOR, the acceleration parameters are calculated in terms of the iterative solution as a part of the algorithm. Moreover, the present method is not strongly influenced by using stretched grids.

There are some indications, that a similar performance can be expected for three-dimensional calculations.

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